Problem 2A,9
Prove that $|A| = \lim_{t\to\infty} |A \cap (-t,t)|$ for all $A \subset \mathbb{R}$.

Proof: By 2.5 we know that the limit exists and $|A| \geq \lim_{t\to\infty} |A \cap (-t,t)|$. Now we prove the converse direction. First note that

$$|A \cap (-t,t)| = |A \cap (-t,t]|,$$

then we have

$$|A| = |\bigcup_{k=-\infty}^{+\infty} A \cap (k,k+1]| \leq \sum_{k=\infty}^{+\infty} |A \cap (k,k+1]| = \lim_{N\to+\infty} \sum_{k=-N}^{+N} |A \cap (k,k+1]| = \lim_{N\to\infty} |A \cap (-N,N+1)|$$

Using

$$\lim_{N\to\infty} |A \cap (-N,N+1)| = \lim_{t\to\infty} |A \cap (-t,t)|$$

then we finish the proof.

Problem 2A,10
Prove that $|[0,1] - \mathbb{Q}| = 1$.

Proof: Note that $|[0,1]| = 1, |\mathbb{Q}| = 0$ by 2.4 and 2.14.Obviously $|[0,1] - \mathbb{Q}| \leq 1$, and on the other hand,

$$1 = |[0,1]| = |\mathbb{Q} \cup ([0,1] - \mathbb{Q})| \leq |\mathbb{Q}| + |[0,1] - \mathbb{Q}| = |[0,1] - \mathbb{Q}|$$

this completes the proof.

Problem 2B,14
a. Suppose $f_1, f_2, \ldots$ is a sequence of functions from a set X to $\mathbb{R}$. Explain why

$$\{x \in X : \textit{the sequence } f_1, f_2, \ldots \textit{has a limt}\} = \bigcap_{n=1}^{\infty} \bigcup_{j=1}^{\infty} \bigcap_{k=j}^{\infty} (f_j - f_k)^{-1}((-\frac{1}{n}, \frac{1}{n}))$$

b. Suppose $(X, \mathcal{S})$ is a measurable space and $f_1, f_2, \ldots$ is a sequence of $\mathcal{S}-$ measurable functions from X to $\mathbb{R}$. Prove that

$$\{x \in X : \textit{the sequence } f_1, f_2, \ldots \textit{has a limt}\}$$

is an $\mathcal{S}$ measurable set.

Proof: For a, we just recall that if $f_1, f_2, \ldots$ has a limit at x, then for any positive integer n, there exists $N = N(n)$ such that for any $j, k > N$, we have $|f_j(x) - f_k(x)| < \frac{1}{n}$. This is exactly the conclusion of a.

For b, just note that the set in the right hand of a is an $\mathcal{S}$ measurable set.

Problem 2B,25
Suppose $B \subset \mathbb{R}$ and $f : B \to \mathbb{R}$ is an increasing function. Prove that there exists a sequence of funcions $f_1, f_2, \ldots$ of strictly increasing functions from B to $\mathbb{R}$ such that for every $x \in B$,

$$f(x) = \lim_{k\to\infty} f_k(x).$$

Proof: $f_k(x) = f(x) + \frac{x}{k}$ satisfies the conclusion.

Problem 2C,1
Explain why there does not exist a measure space $(X, \mathcal{S}, \mu)$ with the property that $\{\mu(E) : E \in S\} = [0,1)$

Proof: Otherwise for such $(X, \mathcal{S}, \mu)$, take $E_i \in \mathcal{S}$ such that $\mu(E_i) = 1 - \frac{1}{i}$. We know that $E = \bigcup_{j=1}^{\infty} E_i$ is also measurable but $|E| \geq \limsup_{j\to\infty} E_j \geq 1$. This is a contradiction with our definition, which completes the proof.

Problem 2C,2
Suppose μ is a measure on $(\mathbb{Z}^+, 2^{\mathbb{Z}^+})$. Prove that there is a sequence $\omega_1, \omega_2, \ldots$ in $[0, \infty]$ such that for every set $E \subset \mathbb{Z}^+$

$$\mu(E) = \sum_{k \in E} \omega_k$$

Proof: Just let $\omega_k = \mu(\{k\})$ for any $k \in \mathbb{Z}^+$.

Problem 2D,5
Prove that if $A \subset \mathbb{R}$ is a Lebesgue measurable, then there exists an increasing sequence $F_1 \subset F_2 \subset \ldots$ of closed sets contained in A such that

$$|A - \bigcup_{k=1}^{\infty} F_k| = 0$$

Proof: By 2.71, we know that for any k, there exists a closed set $E_k \subset A$ such that $|A - E_k| < \frac{1}{k}$. Now take $F_k = \bigcup_{j=1}^{k} E_j$. These F_k satisfy all the condition, which is easy to check.

Problem 2D,12
Suppose $b < C$ and $A \subset (b, c)$. Prove that A is Lebesgue measurable if and only if

$$|A| + |(b, c) - A| = c - b$$

Proof: The "only if" part is obvious. Conversely, from the definition of outer measure we know that there exists a Borel set $B \subset (b, c)$ containing A such that $|B| = |A|$. And we know that

$$|B| + |(b, c) - B| = c - b$$

Note that B i Lebesgue measurable imply

$$|(b, c) - A| = |((b, c) - A) \cap ((b, c) - B)| + |((b, c) - A) \cap B| = |B - A| + |(b, c) - B|$$

so we get that

$$|B - A| = |(b, c) - A| - |(b, c) - B| = |B| - |A| = 0$$

which imply $B - A$ is Lebesgue measurable and $A = B - (B - A)$ is also Lebesgue measurable.

Problem 2D,13
Suppose $A \subset \mathbb{R}$. Prove that A is Lebesgue measurable if and only if for every positive integer n,

$$|(-n, n) \cap A| + |(-n, n) - A| = 2n$$

Proof: If A is Lebesgue measurable, then by 2.70 there exists a Borel set $B \subset A$ such that $|A - B| = 0$. Now by 2.66,

$$|(-n, n) \cap B| + |(-n, n) - B| = 2n$$

Combining with $|A - B| = 0$, we get that

$$|(-n, n) \cap B| + |(-n, n) - B| = 2n$$

Conversely, suppose that

$$|(-n, n) \cap A| + |(-n, n) - A| = 2n$$

for any positive integers n. Using Problem 2D,12 we know that $A \cap (-n, n)$ is Lebesgue measurable. Then $B = \bigcup_{j=1}^{\infty} B \cap (-n, n)$ is also Lebesgue measurable. This completes the proof.